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An analysis of the Krori–Barua solution

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Abstract. The constants of the Krori–Barua metric are explicitly fixed in terms of the physical constants of mass, charge and radius of the source. The conditions for physical relevance are investigated leading to a functional dependence of the ratio of mass-to-radius on the ratio of charge-to-mass and also to upper and lower limits on these ratios. The functional dependences of the physical variables of the problem on the fractional radius, with the square of the charge-to-mass ratio as a parameter, are presented. The information obtained regarding the Krori–Barua solution is illustrated graphically throughout.

1. Introduction

In recent years much interest has been focused on finding interior solutions of the Einstein equations corresponding to static charged spheres. Spheres of charged dust have been investigated by Bonnor and Wickramasuriya (1975) and Raychaudhuri (1975). Interior solutions for charged fluid spheres have been presented by Efinger (1965), Kyle and Martin (1967), Wilson (1967), Kramer and Neugebauer (1971) and, most recently, by Krori and Barua (1975). Of the fluid sphere solutions the latter four are of special interest since, with the imposition of suitable conditions, they are completely free of metric singularities and satisfy physical considerations.

In this paper the Krori and Barua (KB) solution, in particular, is analysed. The constants appearing in the KB paper are evaluated explicitly in terms of the physical constants of mass, charge and radius of the source (see § 2). The restrictions imposed by physical conditions on the KB solution are then investigated in detail, leading to a functional dependence between mass-to-radius ratio and charge-to-mass ratio and also to upper and lower limits on these ratios (see § 3). The functional dependences of pressure, electrostatic energy density, mass density, electric field and charge density on the fractional radius are illustrated in specimen curves for various values of the parameter, charge-to-mass ratio squared (figures 2–4). Some aspects of the curves are discussed in § 4.

2. Fixing the KB constants

Krori and Barua (1975) demonstrated that the spherically symmetric metric

$$ds^2 = -e^{\lambda(r)}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + e^{\nu(r)}\, dt^2 \tag{2.1}$$

with

$$\lambda = Ar^2 \tag{2.2}$$

and

$$\nu = Br^2 + C, \tag{2.3}$$

where A, B and C are constants, satisfies the Einstein–Maxwell equations and corresponds to a charged sphere of perfect fluid with the internal pressure P, the electrostatic energy density E†, the matter density ρ, and the charge density σ, given by the following relations:

$$16\pi P = e^{-Ar^2}[4B - A + B(B-A)r^2 + r^{-2}] - r^{-2} \tag{2.4}$$

$$16\pi E = e^{-Ar^2}[B(B-A)r^2 - A - r^{-2}] + r^{-2} \tag{2.5}$$

$$16\pi\rho = e^{-Ar^2}[5A - B(B-A)r^2 - r^{-2}] + r^{-2} \tag{2.6}$$

and

$$4\pi\sigma = \left(\frac{dF^{41}}{dr} + \frac{2}{r}F^{41} + (A+B)rF^{41}\right) e^{(Br^2+C)/2} \tag{2.7}$$

where

$$F^{41} = \left[e^{-2Ar^2-Br^2-C}\left(\frac{B^2r^2}{2} - \frac{AB}{2}r^2 - \frac{A}{2} - \frac{1}{2r^2}\right) + \frac{e^{-Ar^2-Br^2-C}}{2r^2}\right]^{1/2}. \tag{2.8}$$

Since the exterior metric which is the extension of this KB interior is necessarily the Reissner–Nordström metric (Reissner 1916, Nordström 1918)

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.9}$$

continuity of the first and second fundamental forms across the surface of the fluid sphere implies

$$e^{-AR_b^2} = \left(1 - 2\frac{M}{R_b} + \frac{Q^2}{R_b^2}\right) \tag{2.10}$$

(from continuity of g_{rr} using (2.1), (2.2) and (2.9)),

$$e^{(BR_b^2+C)} = \left(1 - \frac{2M}{R_b} + \frac{Q^2}{R_b^2}\right) \tag{2.11}$$

(from continuity of g_{tt} using (2.1), (2.3) and (2.9)), and

$$B\, e^{BR_b^2+C} = \frac{M}{R_b^3} - \frac{Q^2}{R_b^4} \tag{2.12}$$

(from continuity of $\partial g_{tt}/\partial r$ using (2.1), (2.3) and (2.9)), where M is the mass of the source, Q its charge and R_b its radius. Equations (2.10), (2.11) and (2.12) may be solved for A, B and C yielding

$$A = \left[-\ln\left(1 - \frac{2M}{R_b} + \frac{Q^2}{R_b^2}\right)\right] R_b^{-2} \tag{2.13}$$

$$B = \left(\frac{M}{R_b} - \frac{Q^2}{R_b^2}\right)\left[R_b^2\left(1 - \frac{2M}{R_b} + \frac{Q^2}{R_b^2}\right)\right]^{-1} \tag{2.14}$$

† Throughout this paper the term 'electrostatic energy density' is taken to mean $E \equiv -F^{41}F_{41}/8\pi$.

and

$$C=\ln\left(1-\frac{2M}{R_b}+\frac{Q^2}{R_b^2}\right)-\left[\left(\frac{M}{R_b}-\frac{Q^2}{R_b^2}\right)\left(1-\frac{2M}{R_b}+\frac{Q^2}{R_b^2}\right)^{-1}\right]. \tag{2.15}$$

Equations (2.13)–(2.15) fix the constants A, B and C in terms of the physical constants of mass, charge, and radius of the source.

3. Conditions for physical relevance

The mass, charge and radius of the KB source are not completely independent but are related via a restriction imposed by the condition that the pressure be equal to zero on the surface of the fluid sphere. According to equation (2.4), in conjunction with (2.13) and (2.14), this condition implies

$$\ln(1-2k+qk^2)=f(k,q)\equiv\frac{k(3qk-2-6qk^2+3k+2q^2k^3)}{(1-2k+qk^2)(1-k)} \tag{3.1}$$

where

$$k\equiv M/R_b \tag{3.2}$$

and

$$q\equiv(Q/M)^2\dagger. \tag{3.3}$$

Equation (3.1) is a transcendental equation giving the functional dependence of the ratio of mass-to-radius, k, on the charge-to-mass ratio squared, q. For each q, there exist two physical k possibilities. One is the trivial $k=0$ solution corresponding to a flat space with mass and charge densities equal to zero. (Note that $k\equiv M/R_b=0$ necessarily implies that Q^2/R_b^2 also equals zero since otherwise A, given by equation (2.13), would be negative. This would in turn imply that the central density is negative as can be established from equation (2.6).) The non-trivial physical solutions of equation (3.1), a unique value of k for a given q, are illustrated in figure 1.

In addition to the relation given by equation (3.1), imposed on the ratios k and q by the condition that the surface pressure be equal to zero, there are restrictions dictated by other physical conditions. The central density must be greater than or equal to three times the central pressure. According to equations (2.4) and (2.6) this implies that

$$A\geqslant B \tag{3.4}$$

or, using equations (2.13) and (2.14), in conjunction with the definitions (3.2) and (3.3),

$$-\ln(1-2k+qk^2)\geqslant k(1-qk)/(1-2k+qk^2). \tag{3.5}$$

The restriction that this condition imposes on k and q is denoted in figure 1 by cross-hatching.

The upper limit on the allowable k values, k_+, is determined by the intersection of the curves corresponding to equation (3.1) and the equality statement in condition (3.5), that is by the following system of equalities:

$$-\ln(1-2k_++q_+k_+^2)=-f(k_+,q_+)=k(1-q_+k_+)/(1-2k_++q_+k_+^2) \tag{3.6}$$

† It is assumed that $(1-2k+qk^2)\neq0$ since otherwise either the central mass density or radius of the sphere is infinite.

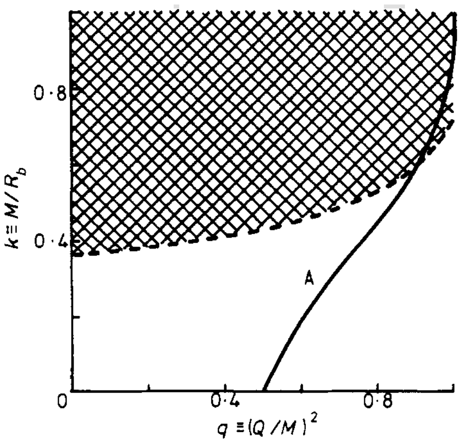

Figure 1. A graph of the non-trivial solutions of equation (3.1), $\ln(1-2k+qk^2)=f(k,q)$ (curve A), where the mass-to-radius ratio, $k \equiv M/R_b$, is plotted against the charge-to-mass ratio squared, $q \equiv (Q/M)^2$. The region of k–q space forbidden by its central density being less than three times its central pressure is denoted by cross-hatching.

where q_+, corresponding to k_+, is the upper limit for the q values. The latter equality in (3.6) implies that

$$k_+ q_+ = \tfrac{1}{2} \tag{3.7}$$

exactly.

The former equality, in conjunction with equation (3.7), implies that

$$k_+ \sim 0{\cdot}567. \tag{3.8}$$

Equations (3.7) and (3.8) yield the upper limit for q,

$$q_+ \sim 0{\cdot}881. \tag{3.9}$$

The physical condition that $k \equiv M/R_b$ be non-negative fixes the lower limits on the k and q values of the non-trivial solutions of equation (3.1) depicted in figure 1. The lower limit for the value of k is

$$k_- = 0 \tag{3.10}$$

whilst that for q is

$$q_- = \tfrac{1}{2}. \tag{3.11}$$

The remaining central and surface conditions imposed by physical considerations are satisfied by the solutions of equation (3.1) under the upper and lower limit restrictions.

The further physical conditions demanding that throughout the sphere's interior the pressure, electrostatic energy density and mass density be positive have all been shown to be satisfied by Krori and Barua. One can similarly demonstrate that the remaining condition

$$\rho \geqslant 3P \tag{3.12}$$

is also satisfied throughout the interior. Using equations (2.4) and (2.6) one obtains

$$16\pi(\rho-3P)=e^{-Ar^2}[8A-4B(B-A)r^2-4r^{-2}-12B+4r^{-2}e^{Ar^2}]$$

$$=2e^{-Ar^2}\left[6(A-B)+r^2[2B(A-B)+A^2]\right.$$

$$\left.+2A^3r^4\left(\frac{1}{3!}+\frac{Ar^2}{4!}+\frac{A^2r^4}{5!}+\dots\right)\right]. \tag{3.13}$$

As a consequence of (3.4) and the relations

$$2B\geqslant A \tag{3.14}$$

and

$$A\geqslant 0 \tag{3.15}$$

which follow, respectively, via equations (2.4) and (2.6), from the conditions that the central pressure and mass density be positive, the right-hand side of equation (3.13) is necessarily positive. This, in turn, implies that the condition (3.12) is satisfied.

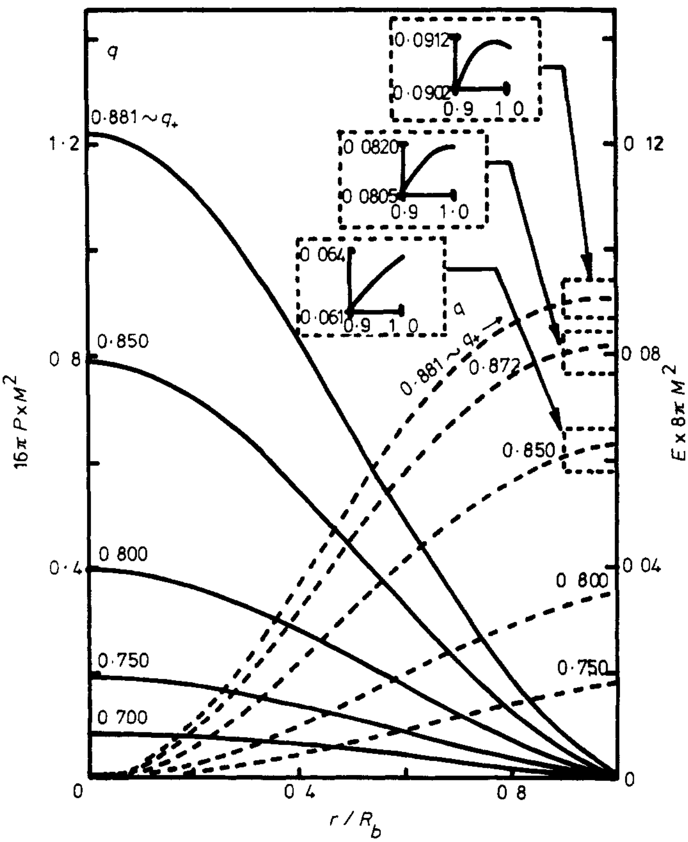

Figure 2. Specimen curves of the pressure P ($\times\,16\pi M^2$) and the electrostatic energy density E ($\times\,8\pi M^2$), against fractional radius r/R_b, for various values (marked on the respective curves) of the parameter q charge-to-mass ratio squared (M = mass of the sphere; R_b = radius of the sphere). Pressure curves are represented by full curves and electrostatic energy density curves by broken curves.

4. The physical variables

As is evident from equations (2.4)–(2.8), the physical variables of pressure, electrostatic energy density, mass density, charge density and electric field are functions of the fractional radius r/R_b, with parametric dependence on the constants A and B or, equivalently by equations (2.13) and (2.14), on the constants k and q. However, as we have seen above, equation (3.1) establishes a functional one-to-one relation between the non-trivial k and q values. Thus, the physical variables in essence depend on only one parameter which is chosen to be q. In figures 2–4 the physical variables are plotted against fractional radius for specific values of this q parameter.

In the graphs of the charge density σ (figure 3) and the electric field F^{41} (figure 4), the sphere is assumed to be positively charged. If on the contrary the sphere is negatively charged, figures 3 and 4 are amended by reflecting the curves in the $\sigma = 0$ and $F^{41} = 0$ axes respectively.

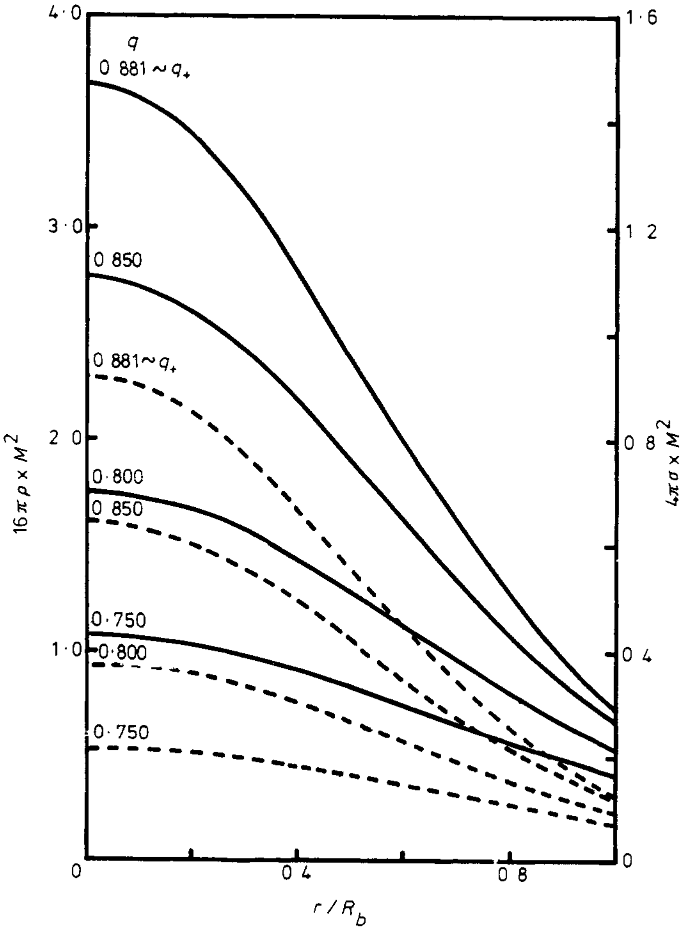

Figure 3. Specimen curves of the mass density ρ ($\times 16\pi M^2$) and the charge density σ ($\times 4\pi M^2$) against fractional radius r/R_b, for various values of the parameter q charge-to-mass ratio squared (M = mass of the sphere; R_b = radius of the sphere). The mass density curves are represented by full curves and the charge density curves by broken curves.

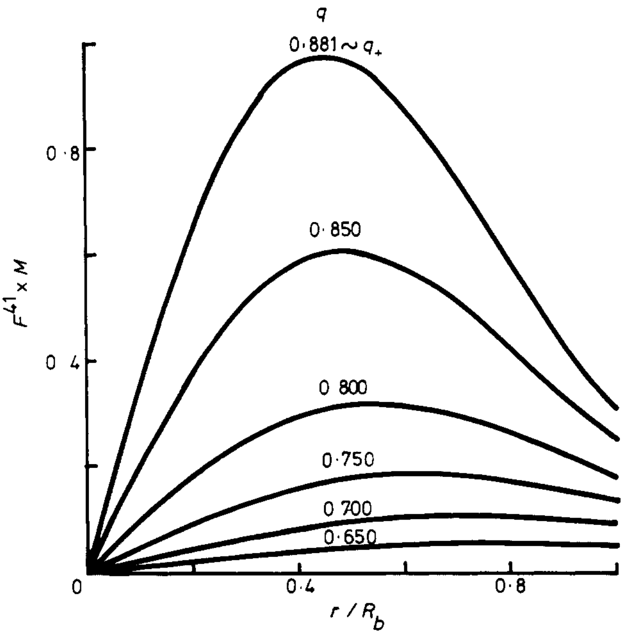

Figure 4. Specimen curves of the electric field F^{41} ($\times M$), against fractional radius r/R_b, for various values of the parameter q charge-to-mass ratio squared (M = mass of the sphere; R_b = radius of the sphere).

It is of interest that, owing to relativistic effects, the maxima of the electrostatic energy density and electric field curves (cf figures 2 and 4) do not coincide as they do in non-relativistic Maxwell theory.

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References

Bonnor W B and Wickramasuriya S B P 1975 *Mon. Not. R. Astron. Soc.* **170** 643–9
Efinger H J 1965 *Z. Phys.* **188** 31–7
Kramer D and Neugebauer G 1971 *Ann. Phys., Lpz.* **27** 129–35
Krori K D and Barua J 1975 *J. Phys. A: Math. Gen.* **8** 508–11
Kyle C F and Martin A W 1967 *Nuovo Cim.* **50** 583–604
Nordström G 1918 *Proc. K. Ned. Akad. Wet.* **20** 1238–45
Raychaudhuri A K 1975 *Ann. Inst. Henri Poincaré* A **22** 229–40
Reissner H 1916 *Ann. Phys., Lpz.* **50** 106–20
Wilson S J 1967 *Can. J. Phys.* **47** 2401–4